MATH 2260 Calculus II
Eric Perkerson

**Name:** Answers
Date: June 30, 2015

# Practice Midterm

Show all of your work, label your answers clearly, and do not use a calculator.

**Problem 1** When the ellipse given by $x^2/3^2+y^2/2^2=1$ is revolved around the horizontal axis we obtain an "ellipsoid of revolution." Find the volume of the enclosed solid. See the below figure, where $a=3$ and $b=2$.

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

$$\Rightarrow \frac{y}{2}=\sqrt{1-\frac{x^2}{9}}$$

$$\Rightarrow y=2\sqrt{1-\frac{x^2}{9}}$$

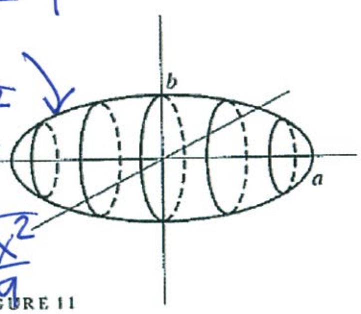



FIGURE 11

$$V=\int_{-3}^{3} A(x)\,dx$$

$$A(x)=\pi r^2=\pi\left(2\sqrt{1-\frac{x^2}{9}}\right)^2$$

$$=4\pi\left(1-\frac{x^2}{9}\right)$$

$$\Rightarrow V=\int_{-3}^{3}4\pi\left(1-\frac{x^2}{9}\right)dx=4\pi\left[x-\frac{x^3}{27}\right]_{-3}^{3}$$

$$=4\pi\left(3-1-(-3+1)\right)$$

$$=4\pi(6-2)$$

$$=16\pi$$

**Problem 2** Find the volume of the torus (see the figure, where $a = 5$ and $b = 3$) obtained by rotating the circle $(x-5)^2 + y^2 = 3^2$ around the vertical axis. Note that in the picture, the vertical axis is the $y$-axis.

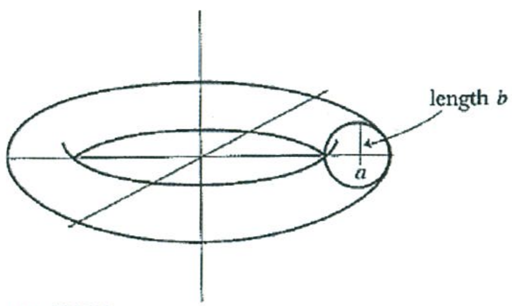



FIGURE 12

See class notes from today (7/6/15).

**Problem 3** Find the length of the curve given by $x = \dfrac{y^3}{3} + \dfrac{1}{4y}$ from $y = 1$ to $y = 6$.

See Webwork Solutions for 6.3.2

**Problem 4** Find the area of the surface generated by revolving the curve $x = \dfrac{e^y + e^{-y}}{2}$ from $0 \leq y \leq \ln(2)$ about the $y$-axis.

See Webwork Solutions 6.4.4

**Problem 5** Solve the differential equation given by

$$\frac{dy}{dt} = \frac{5t}{3t(t-1)ye^y}$$

$$\Rightarrow \int \underbrace{y}_{g}\underbrace{e^y}_{f'}\,dy = \int \frac{5\cancel{t}}{3\cancel{t}(t-1)}\,dt$$

$$\Rightarrow ye^y - \int (1)e^y\,dy = \frac{5}{3}\int \frac{1}{t-1}\,dt$$

$$\Rightarrow ye^y - e^y = \frac{5}{3}\ln(|t-1|) + C$$

So you can't actually solve for $y$ in this case, but expect to on the test.

**Problem 6** A bathroom scale is compressed 1/12 in when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, answer the following:

**a** *What is the scale's force constant? (Remember the appropriate units!)*

See Webwork Solutions

**b** *How much does someone who compresses the scale 1/8 in weigh? (Units!)*

**c** *How much work is done compressing the scale 1/8 in? (Units!)*

**Problem 7** Evaluate the integral

$$\int \underbrace{x \log(x)}_{} dx \quad \cancel{= x\log(x) - x/2}$$

$$\int \underbrace{x}_{f'} \underbrace{\log(x)}_{g} dx = \frac{x^2}{2}\log(x) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^2}{2}\log(x) - \frac{1}{2}\left[\frac{x^2}{2}\right] + C$$

**Problem 8** Evaluate the integral

$$\int 6e^{-y}\cos(y)dy = \frac{1}{2}\left(6e^{-y}\left(\sin(y) - \cos(y)\right)\right) + C$$

$$= 6\int \underbrace{e^{-y}}_{f'} \underbrace{\cos(y)}_{g} dy = 6\left(-e^{-y}\cos(y) - \int(-e^{-y})(-\sin(y))dy\right)$$

$$= -6e^{-y}\cos(y) - 6\int \underbrace{e^{-y}}_{f'} \underbrace{\sin(y)}_{g} dy$$

$$= -6e^{-y}\cos(y) - 6\left(-e^{-y}\sin(y) - \int -e^{-y}\cos(y)dy\right)$$

$$= -6e^{-y}\cos(y) + 6e^{-y}\sin(y) - 6\int e^{-y}\cos(y)dy$$

$$\Rightarrow 2\left(6\int e^{-y}\cos(y)dy\right) = 6\left(e^{-y}\right)\left(\sin(y) - \cos(y)\right) + C$$

**Problem 9** Solve the differential equation

$$\int \sec^3(x)dx$$

See class notes

**Problem 10** Evaluate the integral

Typo

$$\int \cos^2(x)\sin^(x)dx$$

$u = \cos(x)$

$du = -\sin(x)dx$

$$\Rightarrow \int \cos^2(x)\sin(x)dx = \int u\,du = \frac{u^2}{2} + C = \frac{\cos^2(x)}{2} + C$$

---

Original problem was meant to be

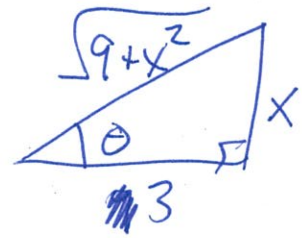
$$\int \cos^2(x)\sin^2(x)\,dx = \int \left(\frac{1+\cos(2x)}{2}\right)\left(\frac{1-\cos(2x)}{2}\right)dx$$

$$= \frac{1}{4}\int 1 - \cos^2(2x)\,dx = \frac{1}{4}\int \sin^2(2x)dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx$$

...

**Problem 11** Evaluate the integral

$$\int \frac{1}{\sqrt{9+x^2}}dx$$




Use trig sub

$x = 3\tan(\theta)$
$dx = 3\sec^2(\theta)d\theta$

$$\Rightarrow \int \frac{1}{\sqrt{9+x^2}}dx = \int \frac{1}{\sqrt{9+9\tan^2(\theta)}}(3\sec^2(\theta))d\theta = \int \frac{\sec^2\theta}{\sec\theta}d\theta$$

$$= \int \sec\theta \, d\theta = \ln(|\sec\theta + \tan\theta|) + C$$

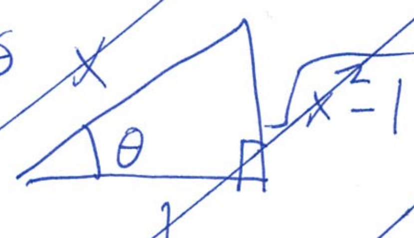
$$= \ln\left(\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right|\right) + C$$

**Problem 12** Evaluate the integral

$$\int \frac{1}{x^2-1}dx$$


Use trig substitution

$$\Rightarrow \int \frac{1}{x^2-1}dx = \int \frac{1}{\sec^2(\theta)-1}(\sec(\theta)\tan\theta)d\theta$$




$$= \int \frac{\sec(\theta)\tan(\theta)}{\tan^2(\theta)}d\theta = \int \frac{\sec(\theta)}{\tan(\theta)}d\theta$$

$x = \sec(\theta)$
$dx = \sec(\theta)\tan(\theta)d\theta$

Use partial fractions instead

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \quad \ldots$$

**Problem 13** Evaluate the integral

$$\int \frac{4x+5}{(x+2)^2}dx$$

Partial fractions : $\dfrac{4x+5}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$

$\vdots$

Final answer : $\dfrac{3}{2+x} + 4\ln(|2+x|) + C$

**Problem 14** Evaluate the integral

$$\int \frac{3x+1}{(x^2+1)^2}dx$$

Partial fractions : $\dfrac{3x+1}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$\vdots$

Final answer : $\frac{1}{2}\left(\dfrac{-3+x}{1+x^2} + \tan^{-1}(x)\right) + C$

**Problem 15** If the integral

$$\int_1^\infty \frac{\cos^2(x)}{x^2}dx$$

converges, give proof by comparison. If it diverges, then prove that by comparison.

$-1 \leq \cos(x) \leq 1$

$\Rightarrow 0 \leq \cos^2(x) \leq 1$

$\Rightarrow 0 \leq \frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}$

$\int_1^\infty \frac{1}{x^2}dx = \lim_{a\to\infty}\left[\frac{-1}{x}\right]_1^a = \lim_{a\to\infty}\left(\frac{1}{a} - \frac{1}{1}\right)$

$= 1$

So both converge.

**Problem 16** If the integral

$$\int_1^\infty \frac{1}{1+x^2}dx$$

converges, prove it using any method you wish. If it diverges, then prove that it diverges.

$\frac{1}{1+x^2} < \frac{1}{x^2}$

Same as above.